SHARP CONSTANTS IN SOME MULTIPLICATIVE SOBOLEV INEQUALITIES *

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Abstract

The optimal constants in the multiplicative Sobolev inequalities where the gradient is estimated in the L_1–norm and the function in two different Lebesgue norms are found. With the optimal constants, these inequalities turn out to still be equivalent to the isoperimetric property of the balls in the Euclidean space. In the course of the proof, relations between Lorentz and Lebesgue spaces are studied (and also applied to some different measures, e.g., Riesz potentials).

1 Introduction

The classical Sobolev inequality for the modulus of the gradient $|\nabla f|$ asserts that

$$\|\nabla f\|_1 \geq c_n \|f\|_\alpha, \tag{1.1}$$

where $\alpha = n/n-1$, $n \geq 2$, where f is an arbitrary smooth function on $\mathbb{R}^n$, and where

$$\|f\|_\alpha = \left(\int_{\mathbb{R}^n} |f(x)|^\alpha dx\right)^{1/\alpha}$$

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is the norm of f in the Lebesgue space $L_\alpha = L_\alpha(\mathbb{R}^n, dx)$, $\|\nabla f\|_1 = \||\nabla f|\|_1$. The optimal constant in (1.1) is 0, if $\alpha \neq n/n-1$, while for $\alpha = n/n-1$, $c_n = n\omega_n^{1/n}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, is optimal (this fact is independently due to Federer and Fleming [FF] and to Maz'ya [M1]). The situation changes considerably, if one is instead interested in finding

$$c_n(\alpha, \beta) = \inf_{\substack{\|f\|_\alpha = 1 \\ \|f\|_\beta = 1}} \|\nabla f\|_1, \tag{1.2}$$

for two different $\alpha, \beta \geq 1$. In this case it is known (Gagliardo [G], Nirenberg [N, p.125]), that $c_n(\alpha, \beta) > 0$ if $\alpha, \beta \leq n/n-1$, or if $\alpha, \beta \geq n/n-1$. We precise this result as follows.

Theorem 1.1 *Let $\alpha, \beta \geq 1$, $\alpha \neq \beta$. Then $c_n(\alpha, \beta) > 0$ if and only if $\alpha, \beta \leq n/n-1$, or $\alpha, \beta \geq n/n-1$, moreover $c_n(\alpha, \beta) = c_n = n\omega_n^{1/n}$.*

Let us see what kind of inequalities are hidden in the variational problem (1.2). Put

$$\mathcal{T}(a, b) = \inf_{\substack{\|f\|_\alpha = a \\ \|f\|_\beta = b}} \|\nabla f\|_1,$$

where the infimum, as in (1.2), is taken over all the smooth (or, equivalently, locally Lipschitz) functions f such that $\|f\|_\alpha = a$, $\|f\|_\beta = b$, $a, b > 0$. Any such f can be written as

$$f(x) = \lambda g(tx), \qquad x \in \mathbb{R}^n,$$

where $\lambda > 0$ and $t > 0$ are chosen in such a way that $\|g\|_\alpha = \|g\|_\beta = 1$, and so are given by

$$\lambda = b^{\frac{\beta}{\beta-\alpha}} a^{-\frac{\alpha}{\beta-\alpha}}, \qquad t = b^{\frac{\alpha\beta}{(\beta-\alpha)n}} a^{-\frac{\alpha\beta}{(\beta-\alpha)n}}.$$

But, $\|\nabla f\|_1 = \lambda \|\nabla g\|_1 / t^{n-1}$, hence $\mathcal{T}(a, b) = \lambda \mathcal{T}(1, 1)/t^{n-1} = \lambda c_n(\alpha, \beta)/t^{n-1}$, or, equivalently,

$$\mathcal{T}(a, b) = c_n(\alpha, \beta) b^{\frac{\beta}{\beta-\alpha} \cdot \left(1 - \alpha \frac{n-1}{n}\right)} a^{\frac{\alpha}{\beta-\alpha} \cdot \left(\beta \frac{n-1}{n} - 1\right)}.$$

Setting $\nu = n/n-1$, Theorem 1.1 can be reformulated as:

Theorem 1.2 *Let $1 \leq \alpha < \beta$. Then, for all smooth functions f on $\mathbb{R}^n$,*

$$\|\nabla f\|_1 \geq c_n \|f\|_\alpha^{\frac{\alpha}{\beta-\alpha}\left(\frac{\beta}{\nu}-1\right)} \|f\|_\beta^{\frac{\beta}{\beta-\alpha}\left(1-\frac{\alpha}{\nu}\right)} \tag{1.3}$$

where the optimal constant $c_n = c_n(\alpha, \beta)$ is positive only when $1 \leq \alpha < \beta \leq \nu$, or when $\nu \leq \alpha < \beta$, and is given by $c_n = n\omega_n^{1/n}$.

Let us, more conveniently, rewrite (1.3) separately for the two cases above:

(i) if $1 \leq \alpha < \beta \leq \nu$, then

$$c_n \|f\|_\beta \leq \|\nabla f\|_1^p \|f\|_\alpha^{1-p}, \qquad p = \frac{\nu(\beta-\alpha)}{\beta(\nu-\alpha)}; \tag{1.4}$$

(ii) if $\nu \leq \alpha < \beta$, then

$$c_n \|f\|_\alpha \leq \|\nabla f\|_1^q \|f\|_\beta^{1-q}, \qquad q = \frac{\nu(\beta-\alpha)}{\alpha(\beta-\nu)}. \tag{1.5}$$

Note that when $\beta = \nu$ in (i) and $\alpha = \nu$ in (ii), (1.4) and (1.5) become the classical Sobolev inequality (1.1).

It immediately follows from the arithmetic–geometric mean inequality, $x^p y^{1-p} \leq px + (1-p)y$, $x, y \geq 0$, $p \in (0,1)$, that:

Corollary 1.1 *(i) If $1 \leq \alpha < \beta \leq \nu$, then*

$$\|f\|_\beta \leq \frac{p}{c_n} \|\nabla f\|_1 + \frac{1-p}{c_n} \|f\|_\alpha; \tag{1.6}$$

(ii) if $\nu \leq \alpha < \beta$, then

$$\|f\|_\alpha \leq \frac{q}{c_n} \|\nabla f\|_1 + \frac{1-q}{c_n} \|f\|_\beta, \tag{1.7}$$

where p and q are respectively defined in (1.4) *and* (1.5).

Remark 1.1 A simple dilation argument (and thanks to the Lebesgue measure) shows that in fact, (1.6) and (1.7) respectively imply (1.4) and (1.5). It should also be noted here that in his famous 1938 paper [S], Sobolev did not actually study the inequality (1.1) but rather (1.6). More precisely, he obtained there (1.6), for $1 \leq \alpha < \beta < \nu = n/n-1$, and with existential constants. Also, with existential constants, (1.6) and (1.7), are due (in greater generality) to Gagliardo [G] and Nirenberg [N] (see for example [N, p.125] with $j = 0$, $m = r = 1$, [Au, pp.93–94] or [M2, 1.4.7]).

Finding the optimal constant in Theorem 1.2 is important to understand the geometric meaning of the multiplicative inequalities (1.3)–(1.5). More precisely, let us approximate the indicator function $f = \mathbf{1}_A$ of a measurable set $A \subset \mathbb{R}^n$ of Lebesgue measure $0 < \mathrm{mes}(A) < +\infty$ and whose boundary is a null set, by smooth functions f_m such that

$$\|f_m\|_\alpha \to \|f\|_\alpha, \qquad \|f_m\|_\beta \to \|f\|_\beta, \qquad \|\nabla f_m\|_1 \to \mathrm{mes}^+(A),$$

where $\mathrm{mes}^+(A)$ denotes the Minkowski content of A. Then, (1.3) becomes

$$\mathrm{mes}^+(A) \geq c_n \left(\mathrm{mes}(A)\right)^{\frac{1}{\beta-\alpha}\left(\frac{\beta}{\nu}-1\right)+\frac{1}{\beta-\alpha}\left(1-\frac{\alpha}{\nu}\right)} = c_n \left(\mathrm{mes}(A)\right)^{1/\nu},$$

and we get the isoperimetric inequality

$$\mathrm{mes}^+(A) \geq c_n \left(\mathrm{mes}(A)\right)^{\frac{n-1}{n}}. \tag{1.8}$$

Thus, the multiplicative inequality (1.3) implies the isoperimetric property of the balls in $\mathbb{R}^n$ which states that, among all the compact sets of a fixed volume, the balls have the least surface measure. For the same reasons, the additive Sobolev inequalities (1.6)–(1.7) imply (1.8), and the constants

$$\frac{p}{c_n}, \qquad \frac{1-p}{c_n}, \qquad \frac{q}{c_n}, \qquad \frac{1-q}{c_n}$$

are optimal. On the other hand, (1.3) will be proved with the help of the isoperimetric inequality (1.8).

2 A Variational Problem in Lorentz Spaces

The standard technique to estimate $\|\nabla f\|_1$ is based on the co–area formula

$$\|\nabla f\|_1 = \int_{-\infty}^{+\infty} \mathrm{mes}^+(\{x \in \mathbb{R}^n : f(x) > t\})dt, \tag{2.1}$$

where f is an arbitrary smooth, compactly supported function on $\mathbb{R}^n$. Combining (1.8) with (2.1) gives

$$\|\nabla f\|_1 \geq c_n \|f\|_{\mathrm{Lor}(\nu)}, \qquad \nu = \frac{n}{n-1}, \tag{2.2}$$

where $\mathrm{Lor}(\nu)$ denotes the Lorentz (Banach) space of measurable functions on $\mathbb{R}^n$ with finite norm

$$\|f\|_{\mathrm{Lor}(\nu)} = \int_0^{+\infty} (\mathrm{mes}\{|f| > t\})^{1/\nu}\, dt. \tag{2.3}$$

Next, the simple inequality $\|f\|_{\mathrm{Lor}(\nu)} \geq \|f\|_\nu$ together with (2.2) gives the Sobolev inequality (1.1) (being based on (2.1) for f smooth, (2.2) extends to all locally Lipschitz functions with compact support). The same approach works in the study of the variational problem (1.2). First note that only non–negative functions f need to be considered, because $\|f\|_\alpha = \||f|\|_\alpha$, and $|\nabla f| \geq |\nabla |f||$ (f is locally Lipschitz). The same reduction applies to the Lorentz space. Therefore, instead of dealing with the notion of modulus of the gradient of a locally Lipschitz function, one can consider the problem of finding

$$d_\nu(\alpha, \beta) = \inf_{\substack{\|f\|_\alpha = 1 \\ \|f\|_\beta = 1}} \|f\|_{\mathrm{Lor}(\nu)}, \tag{2.4}$$

where the infimum is taken over all the non–negative measurable functions f with (or, without) compact support such that $\|f\|_\alpha = \|f\|_\beta = 1$. Moreover, since this problem concerns only distributions of measurable functions, one may assume more generally that we are given a measure space (Ω, μ), and associated to it the Lebesgue spaces $L_\alpha = L_\alpha(\Omega, \mu)$ and the Lorentz spaces $Lor(\nu)$ with norm as in (2.3) with μ instead of mes. Theorems 1.1 and 1.2 will thus follow from:

Theorem 2.1 *Let μ be a positive measure, let $\alpha, \beta \geq 1$, $\alpha \neq \beta$, and let $\nu \geq 1$. Then, $d_\nu(\alpha, \beta) \geq 1$ if $\alpha, \beta \leq \nu$, or if $\alpha, \beta \geq \nu$. Hence, for any (non–negative) measurable function f, we have:*

(i) if $1 \leq \alpha < \beta \leq \nu$, then

$$\|f\|_\beta \leq \|f\|_{\mathrm{Lor}(\nu)}^p \|f\|_\alpha^{1-p}, \qquad p = \frac{\nu(\beta - \alpha)}{\beta(\nu - \alpha)}; \tag{2.5}$$

(ii) if $\nu \leq \alpha < \beta$, then

$$\|f\|_\alpha \leq \|f\|_{\mathrm{Lor}(\nu)}^q \|f\|_\beta^{1-q}, \qquad q = \frac{\nu(\beta - \alpha)}{\alpha(\beta - \nu)}. \tag{2.6}$$

Moreover, when the range of μ *is* $[0,+\infty]$, $d_\nu(\alpha,\beta) = 1$, *if* $\alpha,\beta \le \nu$, *or* $\alpha,\beta \ge \nu$, $d_\nu(\alpha,\beta) = 0$ *otherwise, and the inequalities (2.5) and (2.6) are optimal.*

When the range of μ is $[0,+\infty]$, we will show that (2.4) is attained at a function $f = x\mathbf{1}_A$ where (in unique way)

$$x = b^{\frac{\beta}{\beta-\alpha}} a^{-\frac{\alpha}{\beta-\alpha}}, \quad \mu(A) = b^{-\frac{\alpha\beta}{\beta-\alpha}} a^{\frac{\alpha^2}{\beta-\alpha}}.$$

When the measure μ is finite and/or has atoms, it is possible for such functions f not to exist. So, in general, for the indicated α and β, $d_\nu > 1$ and the inequalities (2.5)–(2.6) hold with better constants (this is, for example, easily seen on the two point space).

A minimization problem related to (2.4) was studied (for finite measures) in [BH]. In particular, it is shown there that such problems (involving Lorentz–type norms) have a solution (a minimizer) taking only finitely many values given by the number of constraints. Let us now explain how, similarly, the extremal functions f in (2.4) take at most three values one of which is 0. In (2.4), one can assume that f takes only finitely many values (if (2.5)–(2.6) hold for such f, they continue hold for all measurable f). So, let f take $m+1$ values $x_1 > \cdots > x_m > x_{m+1} = 0$, $m \ge 2$, and put $p_k = \mu(f = x_k)$, $1 \le k \le m$. The x_k are fixed for now, while $p_k \ge 0$ may vary arbitrarily so that

$$\int f^\alpha d\mu = p_1 x_1^\alpha + \cdots + p_m x_m^\alpha = 1, \tag{2.7}$$

$$\int f^\beta d\mu = p_1 x_1^\beta + \cdots + p_m x_m^\beta = 1. \tag{2.8}$$

Moreover, by the definition of the Lorentz norm,

$$\|f\|_{\mathrm{Lor}(\nu)} \equiv R(p_1,\cdots,p_m; x_1,\cdots,x_m) = \sum_{k=1}^m I_\nu(p_1+\cdots+p_k)(x_k - x_{k+1}),$$

where $I_\nu(p) = p^{1/\nu}$. Let us now minimize the functional R on the positive orthant $p_1,\cdots,p_m \ge 0$ under the two conditions (2.7)–(2.8). The first condition (2.7) determines in this orthant an $m-1$-dimensional simplex K with m extreme points $v_1,\cdots,v_m$. The condition (2.8) is the equation of an hyperplane whose intersection with K is a convex set V. Now, by elementary geometry, any extreme point v of V lies on some (one-dimensional) edge of

K, hence, represents a (convex) mixture of some two points v_i and v_j. Since, for each k, only the kth coordinate of v_k is not 0, the extreme points of V have at most two non–zero coordinates. Note then that R is a concave function of the variables p_k (since I_ν is concave), therefore, the minimal value of R on V is attained at some extreme point of V. Thus, one concludes that there is no loss of generality in taking $m = 2$ in the minimizing problem (2.4). Putting $x = x_1$, $y = y_1$, $p_1 = p$, $p_2 = q$, and now also letting x and y to vary, we obtain that

$$d_\nu(\alpha,\beta) = \inf_{\substack{px^\alpha+qy^\alpha=1\\ px^\beta+qy^\beta=1}} y(p+q)^{\frac{1}{\nu}} + (x-y)p^{\frac{1}{\nu}},\ x \geq y \geq 0,\ p,q \geq 0. \qquad (2.9)$$

We are thus left with the calculus problem of minimizing a function on some (2-dimensional) surface in a 4-dimensional space. This will be studied in the next section and it will be shown that (2.9) is attained at $p = 1$, $x = 1$ and $y = 0$, when $\alpha,\beta \leq \nu$ or $\alpha,\beta \geq \nu$, and hence $d_\nu = 1$. It now just remains to explain how the functional inequalities (2.5)–(2.6) appear from (2.4). As in Section 1, let

$$\mathcal{T}_\nu(a,b) = \inf_{\substack{\|f\|_\alpha=a\\ \|f\|_\beta=b}} \|f\|_{\mathrm{Lor}(\nu)}$$

where the infimum is now over all the non–negative functions f taking finitely many values and such that $\|f\|_\alpha = a$, $\|f\|_\beta = b$, $a,b > 0$. Let f take the values $y_1 > \cdots > y_m > y_{m+1} = 0$, $m \geq 2$, and let $q_k = \mu(f = y_k)$, $1 \leq k \leq m$. Since,

$$\int f^\alpha d\mu = q_1 y_1^\alpha + \cdots + q_m y_m^\alpha = a^\alpha,\ \int f^\beta d\mu = q_1 y_1^\beta + \cdots + q_m y_m^\beta = b^\beta,$$

one can change variables $q_k = tp_k$, $y_k = \lambda x_k$ and choose $t > 0$ and $\lambda > 0$ so that p_k and x_k satisfy (2.7)–(2.8). As easily seen,

$$\lambda = b^{\frac{\beta}{\beta-\alpha}} a^{-\frac{\alpha}{\beta-\alpha}}, \qquad t = b^{\frac{\alpha\beta}{\beta-\alpha}} a^{-\frac{\alpha\beta}{\beta-\alpha}},$$

will do it.

Then, since $R(q_1,\cdots,q_m;y_1,\cdots,y_m) = t^{1/\nu}\lambda R(p_1,\cdots,p_m;x_1,\cdots,x_m)$, we get

$$\mathcal{T}_\nu(a,b) = t^{1/\nu}\lambda\,\mathcal{T}_\nu(1,1) = d_\nu(\alpha,\beta) a^{\frac{\alpha}{\beta-\alpha}\left(\frac{\beta}{\nu}-1\right)} b^{\frac{\beta}{\beta-\alpha}\left(1-\frac{\alpha}{\nu}\right)},$$

that is,

$$\|f\|_{\mathrm{Lor}(\nu)} \geq d_\nu(\alpha,\beta)\|f\|_\alpha^{\frac{\alpha}{\beta-\alpha}\left(\frac{\beta}{\nu}-1\right)}\|f\|_\beta^{\frac{\beta}{\beta-\alpha}\left(1-\frac{\alpha}{\nu}\right)}. \tag{2.10}$$

Rewriting (2.10) we obtain (2.5)–(2.6). Note that when the range of μ is not $[0,+\infty]$, one can imbed (Ω,μ) in a measure space (Ω',μ'), such that the range of μ' is $[0,+\infty]$, such that Ω is a measurable subset of Ω' and such that the restriction of μ' to Ω is μ. Then, since (2.5)–(2.6) hold for (Ω',μ'), they hold for (Ω,μ).

3 Proof of Theorem 2.1

Again, let $I_\nu(p) = p^{1/\nu}$ where $\nu \geq 1$. Let $1 \leq \alpha < \beta$. Set

$$d_\nu(\alpha,\beta) \;=\; \inf\; I_\nu(p)(x-y) + I_\nu(p+q)y, \tag{3.1}$$

where the infimum is over all possible $p, q \geq 0$, $x \geq y \geq 0$, such that

$$\left\{ \begin{array}{lcl} px^\alpha + qy^\alpha &=& 1, \\ px^\beta + qy^\beta &=& 1. \end{array} \right.$$

Note that $x = y$ is a solution to this system of equations only if $x = y = 1$, and $p + q = 1$. Likewise, the assumption $y = 0$ implies that $x = 1$ and $p = 1$. In both cases, the right side of (3.1) is equal to 1. We wish to show that these values of x, y and p, q just described, are in fact extremal in (3.1) when $d_\nu(\alpha,\beta) > 0$, and this will give $d_\nu(\alpha,\beta) = 1$. Next, to find d_ν, one need only consider the case $x > y > 0$, where p and q are the unique solution of the system:

$$p = \frac{y^\alpha - y^\beta}{x^\beta y^\alpha - x^\alpha y^\beta}, \quad q = \frac{x^\beta - x^\alpha}{x^\beta y^\alpha - x^\alpha y^\beta}. \tag{3.2}$$

Recalling that $p, q \geq 0$, it is easy to see that $x > 1 > y$.

Lemma 3.1

$$d_\nu(\alpha,\beta) = \left\{ \begin{array}{ll} 0, & \textit{if } \alpha < \nu < \beta, \\ 1, & \textit{otherwise.} \end{array} \right.$$

Proof. Taking into account (3.2), we consider the expression under the infimum (3.1) as a function of two independent variables $x \in (1, +\infty)$ and $y \in (0, 1)$,

$$u(x,y) \;=\; \left[\frac{y^\alpha - y^\beta}{x^\beta y^\alpha - x^\alpha y^\beta}\right]^{\frac{1}{\nu}} (x-y) + \left[\frac{(x^\beta - x^\alpha) + (y^\alpha - y^\beta)}{x^\beta y^\alpha - x^\alpha y^\beta}\right]^{\frac{1}{\nu}} y.$$

When $\alpha < \nu < \beta$, it immediately follows from this last expression that

$$\lim_{x\to+\infty} \lim_{y\to 0^+} u(x,y) \;=\; 0,$$

hence $d_\nu(\alpha, \beta) = 0$. To prove the second part, we use

Lemma 3.2 *Let $\nu \geq 1$, $0 \leq y \leq x$, $y^\nu \leq b \leq a$. Then,*

$$a - b \leq x^\nu - y^\nu \quad\Longrightarrow\quad a^{\frac{1}{\nu}} - b^{\frac{1}{\nu}} \leq x - y. \tag{3.3}$$

Proof. Putting $a = b + \epsilon$, we note that the function $\phi(b) = (b+\epsilon)^{1/\nu} - b^{1/\nu}$ is non–increasing in $b \geq 0$. Therefore,

$$a^{\frac{1}{\nu}} - b^{\frac{1}{\nu}} = \phi(b) \;\leq\; \phi(y^\nu) = (y^\nu + \epsilon)^{\frac{1}{\nu}} - y \;\leq\; x - y,$$

since, $(y^\nu + \epsilon)^{\frac{1}{\nu}} \leq x$ if and only if $\epsilon \leq x^\nu - y^\nu$. Lemma 3.2 is proved.

Now let us rewrite the inequality $u(x,y) \geq 1$ in the form

$$x - y \;\geq\; \left[\frac{x^\beta y^\alpha - x^\alpha y^\beta}{y^\alpha - y^\beta}\right]^{\frac{1}{\nu}} - \left[\frac{(x^\beta - x^\alpha) + (y^\alpha - y^\beta)}{y^\alpha - y^\beta} y^\nu\right]^{\frac{1}{\nu}} \tag{3.4}$$

and, in order to prove (3.4), let us apply Lemma 3.2 with

$$a = \frac{x^\beta y^\alpha - x^\alpha y^\beta}{y^\alpha - y^\beta}, \quad b = \frac{(x^\beta - x^\alpha) + (y^\alpha - y^\beta)}{y^\alpha - y^\beta} y^\nu.$$

When $a < b$ there is nothing to prove, since then the right–hand side of (3.4) is negative while the left–hand side is positive. So, one can assume $a \geq b$. Furthermore, the condition $b \geq y^\nu$ holds true because the fraction before y^ν

in the definition of b is greater than 1. Thus, Lemma 3.1 will be proved if we verify the condition in (3.3):

$$a - b \;\leq\; x^\nu - y^\nu \quad \Longleftrightarrow$$

$$\frac{[x^\beta y^\alpha - x^\alpha y^\beta] - [(x^\beta - x^\alpha) + (y^\alpha - y^\beta)]y^\nu}{y^\alpha - y^\beta} \;\leq\; x^\nu - y^\nu \quad \Longleftrightarrow$$

$$x^\beta(y^\alpha - y^\nu) \;\leq\; x^\alpha(y^\beta - y^\nu) + x^\nu(y^\alpha - y^\beta). \tag{3.5}$$

(3.5) should be proved for all $x > 1 > y > 0$. Next, two possible cases are considered.

Case 1: $1 \leq \alpha < \beta \leq \nu$. Introducing $t = x^{\beta-\alpha}$, we rewrite (3.5) as

$$t(y^\alpha - y^\nu) \;\leq\; (y^\beta - y^\nu) + t^{\frac{\nu-\alpha}{\beta-\alpha}}(y^\alpha - y^\beta). \tag{3.6}$$

Since $(\nu - \alpha)/(\beta - \alpha) > 1$, the right–hand side of (3.6) is a convex function of t, while the left–hand side is linear. In addition, (3.6) becomes equality at $t = 1$. Hence, to prove (3.6) at all $t \geq 1$, it suffices to show (3.6) for the derivatives at $t = 1$ on both sides:

$$y^\alpha - y^\nu \;\leq\; \frac{\nu - \alpha}{\beta - \alpha}(y^\alpha - y^\beta) \quad \Longleftrightarrow$$

$$(\nu - \alpha)y^\beta \;\leq\; (\nu - \beta)y^\alpha + (\beta - \alpha)y^\nu \quad \Longleftrightarrow$$

$$(\nu - \alpha)s \;\leq\; (\nu - \beta) + (\beta - \alpha)s^{\frac{\nu-\alpha}{\beta-\alpha}}, \tag{3.7}$$

where $s = y^{\beta-\alpha} \in (0, 1)$. Again, the right–hand side of (3.7) is a convex function, and obviously, the derivative of the left-hand side majorizes the derivative of the right-hand side. Therefore, in order to prove (3.7) for all $s \in (0, 1)$, it suffices to check it at the end points $s = 0$ and $s = 1$ which is certainly true since $\nu \geq \beta$.

Case 2: $1 \leq \nu \leq \alpha < \beta$. If $\alpha = \nu$, then (3.5) becomes equality. Let $\alpha > \nu$, let $t = x^{\alpha-\nu}$, and rewrite (3.5) as

$$t^{\frac{\beta-\nu}{\alpha-\nu}}(y^\nu - y^\alpha) \;\geq\; t(y^\nu - y^\beta) - (y^\alpha - y^\beta). \tag{3.8}$$

Again, since $(\beta-\nu)/(\alpha-\nu) > 1$, on the above left–hand side, we have a convex function, while the right–hand function is linear. In addition, (3.8) becomes equality at $t=1$. Therefore, to prove (3.8) for all $t \geq 1$, it suffices to compare the derivatives on both sides at $t=1$. But,

$$\frac{\beta-\nu}{\alpha-\nu}(y^\nu - y^\alpha) \;\geq\; y^\nu - y^\beta,$$

if and only if

$$(\beta-\nu)s \;\leq\; (\beta-\alpha)+(\alpha-\nu)s^{\frac{\beta-\nu}{\alpha-\nu}}, \tag{3.9}$$

where $s = y^{\alpha-\nu} \in (0,1)$. Again, (3.9) holds as well as (3.7).

4 Some Extensions and Remarks

The preceding arguments to obtain multiplicative Sobolev inequalities can be extended without any changes to (Lebesgue–Stieltjes) measures μ on $\mathbb{R}^n$ which satisfy the (isoperimetric type) inequality

$$\mu^+(A) \geq c\,(\mu(A))^{1/\nu}, \tag{4.1}$$

where $\mu^+(A)$ is the μ–Minkowski content of A, where A is a Borel set of finite μ-measure.

Theorem 4.1 *If μ satisfies (4.1), then μ satisfies the multiplicative Sobolev inequalities (1.4)–(1.5) with the constant c instead of c_n.*

It is sometimes necessary, not to consider (4.1) for all A, but rather for a class of A satisfying an additional property. Then, (1.4)–(1.5) will also be true for an appropriate class of functions. For example, if (4.1) holds for all A containing a fixed point a, then (1.4)–(1.5) hold for all smooth, compactly supported f with $f(a)=0$. Here are two further examples of such situations.

Example 4.1. Given $r \geq 1$, consider the $n-$dimensional cone

$$\Omega_n(r) = \left\{(x_1,\cdots,x_n) : \sum_{k=1}^{n-1} x_k^2 < x_n^{2r},\ 0 < x_n < 1\right\},$$

and let μ be the restriction of the Lebesgue measure to $\Omega_n(r)$. Then, μ satisfies (4.1) with $\nu = 1 + 1/r(n-1)$ and some positive $c > 0$ ([M1, p.883]). As a result, we also get multiplicative Sobolev inequalities on the cone $\Omega_n(r)$.

Example 4.2. Let μ_n be a Riesz potential on $\mathbb{R}^n$, i.e., $d\mu_n(x) = |x|^{-(n-1)}dx$. The measure μ_n satisfies the Brunn–Minkowski type inequality

$$\mu_n(A+B) \geq \mu_n(A) + \mu_n(B), \tag{4.2}$$

for all measurable sets A and B containing the origin and such that $A + B = \{a+b : a \in A, b \in B\}$ is also measurable. Indeed, μ_n admits the representation

$$\mu_n(A) = \int \lambda_l(A) d\sigma(l),$$

where λ_l is the Lebesgue measure on the line passing through the origin and through the unit length vector l, and where σ is (up to a constant) the Lebesgue measure on the n–sphere of unit radius. Hence, μ_n is a mixture of Lebesgue measures on the lines containing the origin. Now, the Lebesgue measure on the real line satisfies (4.2), and moreover (4.2) (due to its linear form) remains true for mixtures of measures satisfying it. This gives (4.2) for μ_n.

Now taking for B in (4.2) the ball of radius $h > 0$ and letting $h \to 0$, we get (4.1) with $\nu = 1$ and $c = c_n$, for all sets A containing the origin. Thus, for any smooth, compactly supported function f with $f(0) = 0$,

$$\int_{\mathbb{R}^n} \frac{|\nabla f(x)|}{|x|^{n-1}} dx \geq c_n \int_{\mathbb{R}^n} \frac{|f(x)|}{|x|^{n-1}} dx.$$

Moreover, via (2.6) for $\nu = 1$ (or just by Hölder's inequality) the above right–hand side can be estimated via the norms in $L_\alpha(\mu_n)$ and $L_\beta(\mu_n)$.

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